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Operator Versions of Some Classical Inequalities

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ABSTRACT

Using the theory of connections developed in a paper by Kubo and Ando, operator versions of many classical inequalities are obtained. Finally, it is shown how abstract solidarities, an extension of the Kubo-Ando theory, can be used to obtain operator inequalities. © 1997 Elsevier Science Inc.

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1. INTRODUCTION

Let us consider bounded, linear, and positive (i.e. positive semidefinite) operators on an infinite dimensional Hilbert space. A scalar multiple of the identity operator is denoted by the scalar itself; in particular, 1 is the identity operator. The order relation $A \leqslant B$ means that $B - A$ is positive. That $A_1 \geqslant A_2 \geqslant \cdots$ and A_n converges strongly to A is denoted by $A_n \downarrow A$.

A binary operator σ on the class of positive operators, $(A, B) \to A \sigma B$, is called a *connection* if the following requirements are fulfilled [1]:

(I) $A \leqslant C$ and $B \leqslant D \Rightarrow A \sigma B \leqslant C \sigma D$,
(II) $C(A \sigma B)C \leqslant (CAC) \sigma (CBC)$,
(III) $A_n \downarrow A$ and $B_n \downarrow B \Rightarrow (A_n \sigma B_n) \downarrow A \sigma B$.

A *mean* is a connection with normalization condition

(IV) $1 \sigma 1 = 1$.

The following results are also valid [1]: Every mean σ possesses the property

(IV′) $A \sigma A = A$ for every A.

Every connection σ possesses the property

(I′) $(A \sigma B) + (C \sigma D) \leqslant (A + C) \sigma (B + D)$

The simplest examples of means are

Arithmetic mean: $A \triangledown B = \frac{1}{2}(A + B)$.
Harmonic mean: $A \,!\, B = 2(A^{-1} + B^{-1})^{-1}$.
Geometric mean: $A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ for invertible A and B.

Moreover, weighted versions of these means can also be defined.

Let A, B be invertible and $\lambda \in (0, 1)$ be a real number. Then the arithmetic, geometric, and harmonic means are defined, respectively, by

$$A \triangledown_\lambda B = \lambda A + (1 - \lambda)B,$$

$$A \#_\lambda B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-\lambda}A^{1/2},$$

$$A \,!_\lambda\, B = \left[\lambda A^{-1} + (1 - \lambda)B^{-1}\right]^{-1}.$$

It is well known that the following inequalities are valid:

$$A \,!_\lambda\, B \leqslant A \#_\lambda B \leqslant A \triangledown_\lambda B. \tag{1.1}$$

Of course a connection (or a mean) σ is symmetric if

$$A \sigma B = B \sigma A.$$

The following result is valid [2]: Let $A_1 \leqslant \cdots \leqslant A_n$ (not all equal) and $B_1 \leqslant \cdots \leqslant B_n$ (not all equal) be bounded linear, positive, and invertible operators such that $A_1 = B_1$ and $A_n = B_n$, and let $a_1 \leqslant \cdots \leqslant a_n$ be positive numbers. If σ is a symmetric mean, then

$$\left(\sum_{i=1}^{n-1} a_i \Delta A_i \right) \sigma \left(\sum_{i=1}^{n-1} a_i \Delta B_i \right) \leqslant \sum_{i=1}^{n-1} a_i \Delta(A_i \sigma B_i), \tag{1.2}$$

where Δ is a finite difference, i.e., $\Delta a_i = a_{i+1} - a_i$, $i = 1, \dots, n-1$.

In fact, in [2], we have proved a more general result for two symmetric means.

In this paper, we shall show that (1.2) (indeed, a more general result) and operator versions of some general inequalities are consequences of (I′). Finally, it is shown how abstract solidarities, an extension of the Kubo-Ando theory, can be used to obtain operator inequalities.

2. OPERATOR VERSIONS OF CAUCHY, HÖLDER, AND OTHER CLASSICAL INEQUALITIES

Mathematical induction gives, from (I′):

THEOREM 2.1. *Let A_i, B_i, $i = 1, \dots, n$, be bounded linear, and positive operators, and let σ be a connection. Then*

$$\sum_{i=1}^{n} (A_i \sigma B_i) \leqslant \left(\sum_{i=1}^{n} A_i \right) \sigma \left(\sum_{i=1}^{n} B_i \right). \tag{2.1}$$

In the next examples, we assume that A_i and B_i are invertible.

EXAMPLES.

(1) Cauchy's inequality:

$$\sum_{i=1}^{n} A_i^2 \# B_i^2 \leqslant \left(\sum_{i=1}^{n} A_i^2 \right) \# \left(\sum_{i=1}^{n} B_i^2 \right). \tag{2.2}$$

(2) Hölder's inequality: Let $p, q > 0$ with $p^{-1} + q^{-1} = 1$. Then

$$\sum_{i=1}^{n} A_i^p \#_{1/p} B_i^q \leqslant \left(\sum_{i=1}^{n} A_i^p \right) \#_{1/p} \left(\sum_{i=1}^{n} B_i^q \right). \tag{2.3}$$

(3) Minkowski's inequality:

$$\sum_{i=1}^{n} (A_i + B_i)^{-1} \leqslant \left[\left(\sum_{i=1}^{n} A_i^{-1} \right)^{-1} + \left(\sum_{i=1}^{n} B_i^{-1} \right)^{-1} \right]^{-1} \tag{2.4}$$

Indeed, this last inequality follows by letting σ be the parallel sum, i.e.,

$$A \sigma B = (A^{-1} + B^{-1})^{-1} \quad \text{and} \quad A_i = A_i^{-1}, \quad B_i = B_i^{-1}.$$

THEOREM 2.2. *Let A_i, B_i, $i = 1, \dots, n$, be bounded, linear, and positive operators such that*

$$A_1 - A_2 - \cdots - A_n \geqslant 0 \quad \textit{and} \quad B_1 - B_2 - \cdots - B_n \geqslant 0. \tag{2.5}$$

Then

$$A_1 \sigma B_1 - \sum_{i=2}^{n} A_i \sigma B_i \geqslant \left(A_1 - \sum_{i=2}^{n} A_i \right) \sigma \left(B_1 - \sum_{i=2}^{n} B_i \right). \tag{2.6}$$

Proof. By the substitutions

$$A_1 \to A_1 - A_2 - \cdots - A_n, \qquad B_1 \to B_1 - B_2 - \cdots - B_n$$

(2.1) becomes

$$(A_1 - A_2 - \cdots - A_n) \sigma (B_1 - B_2 - \cdots - B_n) + \sum_{i=2}^{n} A_i \sigma B_i \leqslant A_1 \sigma B_1,$$

i.e. (2.6). ∎

In the following examples A_i and B_i are again invertible.

EXAMPLES.

(4) Aczél's inequality: If

$$A_1^2 - A_2^2 - \cdots - A_n^2 > 0 \quad \text{and} \quad B_1^2 - B_2^2 - \cdots - B_n^2 > 0,$$

then

$$A_1^2 \# B_1^2 - \sum_{i=2}^{n} A_i^2 \# B_i^2 \geqslant \left(A_1^2 - \sum_{i=2}^{n} A_i^2 \right) \# \left(B_1^2 - \sum_{i=2}^{n} B_i^2 \right). \quad (2.7)$$

(5) Popoviciu's inequality: If $p, q > 0$, $p^{-1} + q^{-1} = 1$, and

$$A_1^p - A_2^p - \cdots - A_n^p > 0, \qquad B_1^q - B_2^q - \cdots - B_n^q > 0,$$

then

$$A_1^p \#_{1/p} B_1^q - \sum_{i=2}^{n} A_i^p \#_{1/p} B_i^q \geqslant \left(A_1^p - \sum_{i=2}^{n} A_i^p \right) \#_{1/p} \left(B_1^q - \sum_{i=2}^{n} B_i^q \right). \quad (2.8)$$

(6) Bellman's inequality: If

$$A_1^{-1} - A_2^{-1} - \cdots - A_n^{-1} > 0 \quad \text{and} \quad B_1^{-1} - B_2^{-1} - \cdots - B_n^{-1} > 0,$$

then

$$(A_1 + B_1)^{-1} - \sum_{i=2}^{n} (A_i + B_i)^{-1}$$
$$\geqslant \left[\left(A_1^{-1} - \sum_{i=2}^{n} A_i^{-1} \right)^{-1} + \left(B_1^{-1} - \sum_{i=2}^{n} B_i^{-1} \right)^{-1} \right]^{-1} \quad (2.9)$$

REMARKS.

(1) Note that (2.4) and (2.9) can be given in the following forms:

$$\left(\sum_{i=1}^{n} (A_i + B_i)^{-1} \right)^{-1} \geqslant \left(\sum_{i=1}^{n} A_i^{-1} \right)^{-1} + \left(\sum_{i=1}^{n} B_i^{-1} \right)^{-1} \quad (2.4')$$

and

$$\left((A_1 + B_1)^{-1} - \sum_{i=2}^{n} (A_i + B_i)^{-1}\right)^{-1}$$

$$\leqslant \left(A_1^{-1} - \sum_{i=2}^{n} A_i^{-1}\right)^{-1} + \left(B_1^{-1} - \sum_{i=2}^{n} B_i^{-1}\right)^{-1} \tag{2.9'}$$

Note that the following generalization of (2.4′) is obtained in [3] for positive invertible operators A_{ij} $(i = 1, \dots, n,\ j = 1, \dots, m)$:

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{-1}\right)^{-1} \leqslant \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ij}\right)^{-1}\right]^{-1}. \tag{2.10}$$

We can use (2.10) in the proof of the following extension of (2.9′): If positive invertible operators A_{ij} $(i = 1, \dots, m,\ j = 1, \dots, n)$ satisfy the conditions

$$A_{1j}^{-1} - A_{2j}^{-1} - \cdots - A_{mj}^{-1} > 0, \qquad j = 1, \dots, n,$$

then

$$\left(\sum_{j=1}^{m} A_{1j}\right)^{-1} - \sum_{i=2}^{m} \left(\sum_{j=1}^{n} A_{ij}\right)^{-1} \geqslant \left[\sum_{j=1}^{n} \left(A_{1j}^{-1} - \sum_{i=2}^{m} A_{ij}^{-1}\right)^{-1}\right]^{-1} \tag{2.11}$$

(2) A simpler form of Hölder's inequality is

$$\sum_{i=1}^{n} A_i \#_\alpha B_i \leqslant \left(\sum_{i=1}^{n} A_i\right) \#_\alpha \left(\sum_{i=1}^{n} B_i\right), \tag{2.3'}$$

where $0 \leqslant \alpha \leqslant 1$. Setting $A_i \to A_i^s$, $B_i \to A_i^r$ in (2.3′), we get

$$\sum_{i=1}^{n} A_i^{\alpha s + (1-\alpha) r} \leqslant \left(\sum_{i=1}^{n} A_i^s\right) \#_\alpha \left(\sum_{i=1}^{n} A_i^r\right)$$

$$\leqslant \alpha \left(\sum_{i=1}^{n} A_i^s\right) + (1 - \alpha) \left(\sum_{i=1}^{n} A_i^r\right),$$

where we have used (1.1). In fact, we have proved that the function

$$x \mapsto \sum_{i=1}^{n} A_i^x$$

is convex.

3. AN OPERATOR VERSION OF PÓLYA'S INEQUALITY

THEOREM 3.1. *Let $A_1 \leqslant \cdots \leqslant A_n$ (not all equal) and $B_1 \leqslant \cdots \leqslant B_n$ (not all equal) be bounded, linear, and positive operators, let $a = \{a_1, \ldots, a_n\}$ be a nondecreasing positive n-tuple of real numbers, and let σ be a connection. Then*

$$\left(\sum_{i=1}^{n-1} a_i \, \Delta A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i \, \Delta B_i\right) \leqslant \sum_{i=1}^{n} a_i \, \Delta(A_i \, \sigma \, B_i). \tag{3.1}$$

If a is a nonincreasing positive n-tuple of real numbers and $A_1 = B_1 = 0$, then a reverse inequality in (3.1) *holds.*

Proof.

(1) We have

$$\begin{aligned}
&\sum_{i=1}^{n-1} a_i \, \Delta(A_i \, \sigma \, B_i) \\
&\quad = a_n(A_n \, \sigma \, B_n) - a_1(A_1 \, \sigma \, B_1) - \sum_{i=2}^{n} (A_i \, \sigma \, B_i) \, \Delta a_{i-1} \\
&\quad = a_n(A_n \, \sigma \, B_n) - a_1(A_1 \, \sigma \, B_1) - \sum_{i=2}^{n} [(A_i \, \Delta a_{i-1}) \, \sigma \, (B_i \, \Delta a_{i-1})] \\
&\quad \geqslant (a_n A_n) \, \sigma \, (a_n B_n) - (a_1 A_1) \, \sigma \, (a_1 B_1) \\
&\qquad - \left(\sum_{i=2}^{n} A_i \, \Delta a_{i-1}\right) \sigma \left(\sum_{i=2}^{n} B_i \, \Delta a_{i-1}\right) \qquad \text{[by (2.1)]}
\end{aligned}$$

$$\geqslant \left(a_n A_n - a_1 A_1 - \sum_{i=2}^{n} A_i\,\Delta a_{i-1}\right) \sigma \left(a_n B_n - a_1 B_1 - \sum_{i=2}^{n} B_i\,\Delta a_{i-1}\right)$$

[by (2.6) for three terms]

$$= \left(\sum_{i=1}^{n-1} a_i\,\Delta A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i\,\Delta B_i\right).$$

(2) Moreover if a is nonincreasing and $A_1 = B_1 = 0$, we have, since $-a$ is nondecreasing, as a consequence of (2.1),

$$\sum_{i=1}^{n-1} a_i\,\Delta(A_i\,\sigma\,B_i)$$

$$= a_n(A_n\,\sigma\,B_n) + \sum_{i=2}^{n} (A_i\,\sigma\,B_i)\,\Delta(-a_{i-1})$$

$$= (a_n A_n)\,\sigma\,(a_n B_n) + \sum_{i=2}^{n} [A_i\,\Delta(-a_{i-1})]\,\sigma\,[B_i\,\Delta(-a_{i-1})]$$

$$\leqslant (a_n A_n)\,\sigma\,(a_n B_n) + \left(\sum_{i=2}^{n} A_i\,\Delta(-a_{i-1})\right) \sigma \left(\sum_{i=2}^{n} B_i\,\Delta(-a_{i-1})\right)$$

$$\leqslant \left(a_n A_n + \sum_{i=2}^{n} A_i\,\Delta(-a_{i-1})\right) \sigma \left(a_n B_n + \sum_{i=2}^{n} B_i\,\Delta(-a_{i-1})\right)$$

$$= \left(\sum_{i=1}^{n-1} a_i\,\Delta A_i\right) \sigma \left(\sum_{i=1}^{n-1} a_i\,\Delta B_i\right). \qquad \blacksquare$$

REMARK. Theorem 3.1 gives the inequality (1.2) for arbitrary connections (not only symmetric means) and also without the conditions $A_1 = B_1$ and $A_n = B_n$. Also, we have a converse result in our theorem.

COROLLARY 3.1. *Let $A_1 \leqslant \cdots \leqslant A_n$ (not all equal) and $B_1 \leqslant \cdots \leqslant B_n$ (not all equal) be bounded, linear, positive, and invertible operators, and let $\lambda \in [0, 1]$. If a is a nondecreasing n-tuple of positive numbers, then*

$$\left(\sum_{i=1}^{n-1} a_i\,\Delta A_i\right) \#_\lambda \left(\sum_{i=1}^{n-1} a_i\,\Delta B_i\right) \leqslant \sum_{i=1}^{n-1} a_i(A_i\,\#_\lambda\,B_i) \tag{3.2}$$

and

$$\left(\sum_{i=1}^{n-1} a_i\,\Delta A_i\right) !_\lambda \left(\sum_{i=1}^{n-1} a_i\,\Delta B_i\right) \leqslant \sum_{i=1}^{n-1} a_i(A_i\,!_\lambda\,B_i) \tag{3.3}$$

If a is a nonincreasing n-tuple of positive numbers and $A_1 = B_1 = 0$, then reverse inequalities in (3.2) *and* (3.3) *are valid.*

REMARKS. Inequalities (3.2) and (3.3) were proved in [2], but with the additional conditions $A_1 = B_1$ and $A_n = B_n$.

4. INEQUALITIES FOR SOLIDARITIES

An extension of the Kubo-Ando theory was given by J. I. Fujii, M. Fujii, and Y. Seo [4].

A binary operation s on positive operators is an *abstract solidarity* if it satisfies, assuming the existence of $A\,s\,B$ as a bounded operator, that

(S1) $B \leqslant C$ implies $A\,s\,B \leqslant A\,s\,C$,
(S2r) $B_n \downarrow B$ implies $A\,s\,B_n \downarrow A\,s\,B$,
(S2ℓ) $A_n \to A$ strongly implies $A_n\,s\,1 \to A\,s\,1$ strongly, and
(S3) $T^*(A\,s\,B)T \leqslant T^*AT\,s\,T^*BT$.

The solidarity s is superadditive in that

$$(A + B)\,s\,(C + D) \geqslant A\,s\,C + B\,s\,D. \tag{4.1}$$

Of special interest is the relative operator entropy $S(A \mid B)$ for invertible A, B defined by

$$S(A|B) = A^{1/2}(\log A^{-1/2}BA^{-1/2})A^{1/2}$$

Of course, using (4.1), we can prove the following by mathematical induction: Let A_i, B_i, $i = 1, \ldots, n$, be positive operators; then

$$\sum_{i=1}^{n} (A_i\,s\,B_i) \leqslant \left(\sum_{i=1}^{n} A_i\right) s \left(\sum_{i=1}^{n} B_i\right). \tag{4.2}$$

Also, if (2.5) holds, then

$$A_1\,s\,B_1 - \sum_{i=2}^{n} A_i\,s\,B_i \geqslant \left(A_1 - \sum_{i=2}^{n} A_i\right) s \left(B_1 - \sum_{i=2}^{n} B_i\right). \tag{4.3}$$

For the operator entropy, we have

$$\sum_{i=1}^{n} S(A_i|B_i) \leqslant S\left(\sum_{i=1}^{n} A_i \,\middle|\, \sum_{i=1}^{n} B_i\right) \tag{4.4}$$

and, if (2.5) holds,

$$S(A_1|B_1) - \sum_{i=2}^{n} S(A_i|B_i) \geqslant S\left(A_1 - \sum_{i=2}^{n} A_i \middle| B_1 - \sum_{i=2}^{n} B_i\right). \tag{4.5}$$

As in Theorem 3.1, we can also prove

THEOREM 4.1. *Let $A_1 \leqslant \cdots \leqslant A_n$ (not all equal) and $B_1 \leqslant \cdots \leqslant B_n$ (not all equal) be bounded, linear, and positive operators, let $a = \{a_1, \dots, a_n\}$ be a nondecreasing n-tuple of positive numbers, and let s be an abstract solidarity. Then*

$$\left(\sum_{i=1}^{n-1} a_i \, \Delta A_i\right) s \left(\sum_{i=1}^{n} a_i \, \Delta B_i\right) \leqslant \sum_{i=1}^{n-1} a_i \, \Delta(A_i \, s \, B_i). \tag{4.6}$$

If a is a nonincreasing n-tuple of positive numbers and $A_1 = B_1 = 0$, then a reverse inequality in (4.6) is valid.

In the case of operator entropy, (4.6) becomes

$$S\left(\sum_{i=1}^{n-1} a_i \, \Delta A_i \middle| \sum_{i=1}^{n-1} a_i \, \Delta B_i\right) \leqslant \sum_{i=1}^{n-1} a_i S(A_i|B_i). \tag{4.7}$$

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